2-11

Call payoff: $C_N = (S_N - K)^+$

price: $C_n = \widetilde{E}_n\left[\frac{C_N}{(1+r)^{N-n}}\right]$, $n = 0, 1, \cdots, N-1$.

Put payoff: $P_N = (K - S_N)^+$

price: $P_n = \widetilde{E}_n\left[\frac{P_N}{(1+r)^{N-n}}\right]$, $n = 0, 1, \cdots, N-1$

Forward Contract: $F_N = S_N - K$

price: $F_n = \widetilde{E}_n\left[\frac{F_N}{(1+r)^{N-n}}\right]$, $n = 0, 1, \cdots, N-1$.

(i)

$$F_N + P_N = S_N - K + (K - S_N)^+ = (S_N - K)^+ = C_N$$

since $(S_N - K)^+ - (K - S_N)^+ = S_N - K$

(ii)

$$C_n = \widetilde{E}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] = \widetilde{E}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] = \widetilde{E}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \widetilde{E}\left[\frac{P_N}{(1+r)^{N-n}}\right]$$

$$= F_n + P_n.$$

(iii) Discount stock price is martingale:

$$\frac{S_n}{(1+r)^n} = \widetilde{E}_n\left[\frac{S_N}{(1+r)^N}\right].$$

then

$$F_0 = \widetilde{E}_0\left[\frac{F_N}{(1+r)^N}\right] = \widetilde{E}_0\left[\frac{S_N - K}{(1+r)^N}\right] = \widetilde{E}_0\left[\frac{S_N}{(1+r)^N}\right] - \widetilde{E}_0\left[\frac{K}{(1+r)^N}\right]$$

$$= S_0 - \frac{K}{(1+r)^N}$$

(iv) Use your money F_0 to buy one share of stock at price S_0,

then at time N: $(F_0 - S_0)(1+r)^N + S_N = -\frac{K}{(1+r)^N} \cdot (1+r)^N + S_N = -K + S_N =$

(v). By (iii)

$$F_0 = S_0 - \frac{K}{(1+r)^N} = S_0 - \frac{(1+r)^N S_0}{(1+r)^N} = 0.$$

Forward price $= K$.

By (ii) $C_0 = F_0 + P_0 = P_0$.

(vi)

$$F_n = \tilde{E}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] = \tilde{E}_n\left[\frac{S_N - K}{(1+r)^{N-n}}\right] = \tilde{E}_n\left[\frac{S_N}{(1+r)^{N-n}}\right] - \frac{K}{(1+r)^{N-n}}$$

$$= S_n - \frac{(1+r)^N S_0}{(1+r)^{N-n}} = S_n - S_0(1+r)^n \quad \text{not necessarily } 0.$$

then $C_n = P_n$ is not always true. $n \geq 1$.

2.17.

Contract, receive call/put at time m, K.

At time m, the chooser option price is $V_m = \max(C_m, P_m)$.

$$C_m = \tilde{E}_m\left[\frac{(S_N - K)^+}{(1+r)^{N-m}}\right], \quad P_m = \tilde{E}_m\left[\frac{(K - S_N)^+}{(1+r)^{N-m}}\right],$$

And $V_0 = \tilde{E}_0\left[\frac{V_m}{(1+r)^m}\right]$

By put-call parity, since $F_m = S_m - \frac{K}{(1+r)^{N-m}}$,

then $C_m = F_m + P_m = S_m - \frac{K}{(1+r)^{N-m}} + P_m$ and

$$V_m = \max(C_m, P_m) = P_m + \left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+$$

then

$$V_0 = \tilde{E}_0\left[\frac{P_m}{(1+r)^m}\right] + \tilde{E}_0\left[\frac{\left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+}{(1+r)^m}\right]$$

$$= \underbrace{\tilde{E}_0\left[\frac{(K - S_N)^+}{(1+r)^N}\right]}_{\text{put with } K \text{ at } N} + \underbrace{\tilde{E}_0\left[\frac{\left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+}{(1+r)^m}\right]}_{\text{call with } \frac{K}{(1+r)^{N-m}} \text{ at } m}.$$